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Topological foundations of the theory of distibutions

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The following abbreviations are used:
l.t.s. = linear topological space, n.l.s. = normed linear space, s.n.s. = sequentially normed space, u.s.n.s. = union of sequentially normed spaces, c.s. = conjugate space.

Some important theorems

6. Sequentially normed spaces

We consider a linear space X with two different norms $\|x\|_1$ and $\|x\|_2$.

The norms are said to be comparable, with $\|x\|_1$ being the weaker norm and $\|x\|_2$ the stronger norm, if there exists a fixed constant C such that

$$(6.1) \qquad \|x\|_1 \leq C\,\|x\|_2$$

for all $x \in X$.

If X is complete with respect to both norms then it follows from Banach's theorem 5.4 of the inverse operator that comparable norms are even equivalent i.e. there are constants C' and C'' such that

$$(6.2) \qquad \|x\|_1 < C' \,\|x\|_2 < C'' \,\|x\|_1 .$$

If X is not complete for both norms then the process of completion may be applied for either norm. This gives two complete spaces X_1 and X_2.
If $\|x\|_2$ is the stronger norm then any fundamental sequence in X_2 is also a fundamental sequence in X_1. Hence to any element of X_2 there corresponds a unique element of X_1. It is possible that different elements of X_2 correspond to one and the same element of X_1. In order to exclude this the norms are required to be concordant inthe following sense.

The norms $\|x\|_1$ and $\|x\|_2$ of the linear space X are said to be concordant if for every sequence which is fundamental for both norms the convergence to zero for one norm implies that for the other norm.

Example We consider the space of all functions $f(x)$, $0 \leq x \leq 1$, which have a continuous derivative.
The norms

$$\|f\|_1 = \max |f(x)| \qquad \|f\|_2 = \max \{ |f(x)| + |f'(x)| \}$$

are concordant.
However, the norms

$$\|f\|_1 = \max |f(x)| \qquad \|f\|_2 = \max \{ |f(x)| + |f'(0)| \}$$

are not concordant. We may take a sequence $f_n(x)$ with $f_n'(0)=1$ and which converges uniformly to zero. This sequence is a fundamental sequence for both norms which converges to zero for the weaker norm but which does not for the stronger norm since $\|f_n\|_2 \geq 1$ for all n.

In the case of concordant comparable norms we have a one-to-one correspondence between the completion X_2 and a part of the completion X_1 i.e.

$$X_1 \supset X_2 \supset X.$$

From two concordant norms $\|x\|_1$ and $\|x\|_2$ which are not comparable we may easily derive a set of concordant comparable norms by introducing the third norm $\|x\|_3 = \max\{\|x\|_1, \|x\|_2\}$.

We shall now introduce the concept of a sequentially normed space which is of the utmost importance for the theory of generalized functions.
We consider a linear space X with a countable system of norms $\|x\|_1, \|x\|_2, \ldots, \|x\|_m, \ldots$.
In this space a topology is introduced by defining a system of neighbourhoods of zero $U(m,\varepsilon)$ as

(6.3) $\|x\|_1 < \varepsilon, \quad \|x\|_2 < \varepsilon, \quad \ldots, \quad \|x\|_m < \varepsilon$.

One may easily verify that by this choice X becomes a linear topological space which satisfies the first axiom of countability.

The linear space X with a countable system of concordant norms with the above given topology is said to be a <u>sequentially normed space</u>.

The topology of a s.n.s. implies the following definition of convergence. The sequence x_n converges to zero if $\|x_n\|_m \to 0$ for each individual m. Similarly the sequence x_n is a fundamental sequence if for each m and every $\varepsilon > 0$ there is a number $N(m,\varepsilon)$ such that for $k,l > N$ we have $\|x_k - x_l\|_m < \varepsilon$.

We may always assume that the sequence of concordant norms is arranged in order of increasing strength:

(6.4) $\|x\|_1 \leq \|x\|_2 \leq \ldots \leq \|x\|_m \leq \ldots$

In a s.n.s. X with (6.4) the process of completion may be carried outwith respect to each norm. In this way a nesting sequence of Banach spaces

$$X_1 \supset X_2 \supset \ldots \supset X_m \supset \ldots \supset X$$

is obtained. It is to be expected that the limes of X_m, i.e. the intersection of all X_m, gives the completion of X with respect to the topology (6.3).
In fact, we have

Theorem 6.1

The s.n.s. X is complete if and only if it coincides with the intersection of all individual completions X_m.

Proof

1^o If $X = \lim X_m$ we take a fundamental sequence $x_n (n=1,2,\ldots)$ in X and show that it has a limit in X. The sequence x_n is by definition fundamental in each X_m and has there a limit $x^{(m)}$. The concordance of the norms says that all limits $x^{(m)}$ $(m=1,2,\ldots)$ are essentially one and the same limit x which therefore belongs to X. Since now $\|x_n - x\|_m \to 0$ for each m the element x is the limit of the sequence x_n in the topology of X so that X is complete.

2^o If X is complete we take an arbitrary element x of $\lim X_m$ and show that $x \in X$.

There exists for each number m an element $x_m \in X$ for which $\|x - x_m\|_m < \frac{1}{m}$, for X_m is the completion of X with respect to the m^{th} norm.

It is now easy to see that the sequence x_m converges to x for each individual norm, hence $x_m \to x$ in the topology of X. In fact for the p^{th} norm and for $m > p$

$$\|x - x_m\|_p \leq \|x - x_m\|_m < 1/m,$$

so that $\lim_{m\to\infty} \|x - x_m\|_p = 0$ for arbitrary p.

It follows that x_m is a fundamental sequence in X so that the completeness of X implies $x \in X$.

From now on any s.n.s. will be tacitly assumed to be complete.

Example

In the space K(a) of all infinitely differentiable functions $\varphi(x)$ (cf. section 3) a sequence of concordant norms may be introduced by means of

$$\|\varphi_m\| = \max_{|x| \leq a} \{ |\varphi(x)|, |\varphi'(x)|, \ldots, |\varphi^{(m)}(x)| \},$$

$m=0,1,2,\ldots$.

The zero neighbourhoods $U(m,\varepsilon)$, $\|\varphi\|_m < \varepsilon$, coincide with those introduced previously in section 3 and determine consequently the same topology.
That the norms are concordant follows from the fact that if $\varphi, \varphi', \dots, \varphi^{(m)}$ uniformly converge to zero in some interval $(-a,a)$ also $\varphi^{(m+1)}$ converges to zero.

It can be shown by means of the preceding theorem that $K(a)$ is complete. Let $K_m(a)$ denote the space of all functions vanishing outside $|x| \leq a$ which have continuous derivatives up to the m^{th} order and let $\overline{K(a)}^m$ mean the completion of $K(a)$ with respect to the norm m. Then it is clear that

$$\overline{K(a)}^m \subset K_m(a).$$

There remains to prove the opposite inclusion. According to Weierstrass' theorem any function $\varphi(x) \in K_m(a)$ can be approximated by polynomials $P_n(x)$ in such a way that $P_n(x) \to \varphi(x)$ uniformly in x. Take a function $e(x) \in K(a)$ which equals 1 in the interval $|x| \leq a-\delta$ with δ sufficiently small positive. Then $e(x)\, P_n(x)$ converges to $\varphi(x)$ in the topology of $\overline{K(a)}^m$ so that $\varphi(x) \in \overline{K(a)}^m$. This means

$$K_m(a) \subset \overline{K(a)}^m$$

so that

$$K_m(a) = \overline{K(a)}^m$$

Since $K(a)$ is the intersection of all $K_m(a)$ it is complete.

<u>Example</u>
In the space S of all infinitely differentiable functions $\varphi(x)$ for which

$$|x|^k \quad \varphi^{(n)}(x) \to 0 \qquad |x| \to \infty$$

for all k and n $(k=0,1,2,\dots;\ n=0,1,2,\dots)$
a sequence of concordant norms may be introduced by means of

$$\|\varphi\|_m = \max_{k,n \leq m} |x^k \quad \varphi^{(n)}(x)| \ . \qquad m=0,1,2,\dots$$

It can be shown in a similar way as before that S is a (complete) s.n.s.

A s.n.s. may be considered as a linear metric space by defining the distance $d(x,y)$ as

(6.5) $$d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\|x-y\|_m}{1+\|x-y\|_m} .$$

The verification of the six axioms of the linear metric space (cf. section 4) is left to the reader. We note the important fact that the natural topology of the linear metric space is equivalent to the topology of the s.n.s.

In fact, the system of zero neighbourhoods $V(\delta)$ is according to (6.5) determined by spheres

(6.6) $$\sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\|x\|_m}{1+\|x\|_m} < \delta$$

a. Each $U(m,\varepsilon)$ contains a $V(\delta)$ for (6.6) implies

$$\frac{1}{2^k} \frac{\|x\|_k}{1+\|x\|_k} < \delta \qquad \text{for all } k$$

or

$$\|x\|_k < \frac{2^k\delta}{1-2^k\delta}$$

If δ is sufficiently small for $k=1,2,\dots,m$ we shall have $\|x\|_k < \varepsilon$.

b. Each $V(\delta)$ contains a $U(m,\varepsilon)$ for it is possible to determine m and ε such that

$$\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{\|x\|_k}{1+\|x\|_k} < \frac{\varepsilon}{1+\varepsilon} \sum_{k=1}^{m} \frac{1}{2^k} + \sum_{k=m+1}^{\infty} \frac{1}{2^k} < \delta$$

From the equivalence of these topologies it follows that a complete s.n.s. is also a complete metric space. This fact makes it possible to apply the important theorem 4.1 of Baire and its consequence theorem 4.2.

We consider in a linear space X two sequences of norms

$$\|x\|_1 \leq \|x\|_2 \leq \dots \leq \|x\|_m \leq \dots ,$$
$$\|x\|_1' \leq \|x\|_2 \leq \dots \leq \|x\|_m' \leq \dots .$$

Then the first sequence is said to be weaker and the second to be stronger if each norm $\|x\|_k$ $(k=1,2,\dots)$ is weaker than some norm $\|x\|_{m_k}'$ of the second sequence.

If this is the case then each sequence of elements $x_1, x_2, \dots$ which converges with respect to the second system of norms converges also with respect to the first system of norms. We shall show that also the converse is true.

If each sequence $x_1, x_2, \dots$ which converges for all norms $\|x\|'_k$ $(k=1,2,\dots)$ also converges for every norm $\|x\|_k$ $(k=1,2,\dots)$ then the first system is weaker than the second system.

If the proposition is not true there exists a norm $\|x\|_m$ which is not weaker than any norm $\|x\|'_n$ $(n=1,2,\dots)$. Hence, for each n we may find an element x_n for which $\|x_n\|_m > n\,\|x_n\|'_n$. By scalar multiplication, it can be arranged that $\|x_n\|_m = 1$ so that $\|x_n\|'_n < 1/n$. The sequence x_n $(n=1,2,\dots)$ clearly converges to zero for each norm of the second system. However, this would imply convergence to zero for all norms of the first system which contradicts the assumption $\|x_n\|_m = 1$ for all n.

If convergence to zero in the first system implies convergence to zero in the second system and vice versa then the systems are said to be equivalent. In view of the preceding argument a necessary and sufficient condition for equivalence is that each norm of one system is weaker than some norm of the other system.

The definition of a bounded set follows that of boundedness in a more general linear topological space.
The set S in a s.n.s. is bounded if and only if

$$\|x\|_m < C_m \quad (m=1,2,\dots) \text{ for all } x \in S.$$

The notion of a bounded set in a s.n.s. is essentially different from that in an ordinary normed space. For example in a normed space the unit sphere $S(\|x\| \leq 1)$ is bounded and the sets nS $(n=1,2,\dots)$ cover the whole space. On the contrary in a s.n.s. in general there exists no bounded set S for which the sets nS $(n=1,2,\dots)$ cover the whole space. In fact, if S is determined by $\|x\|_m < C_m$ $(m=1,2,\dots)$ then an element x_0 for which $\|x_0\|_m > mC_m$ is not contained in any set nS. That such an element actually exists will be proved below.

Theorem 6.2

Let X be a s.n.s. with a sequence (6.4) of norms and let X_m be the completion with respect to the m^{th} norm. Then if all X_m $(m=1,2,\dots)$ are different it is possible to find for any sequence of positive numbers $c_1, c_2, \dots$ an element $x \in X$ for which

$$\|x\|_m \geq c_m \qquad (m=1,2,\dots),$$

Proof

We shall first show that for each integer m and any choice of positive numbers c and c' it is possible to find an element x for which

$$\|x\|_m \leqq c, \quad \|x\|_{m+1} \geqq c'.$$

In fact, if such an element cannot be found then $\|x\|_m \leqq c$ always implies $\|x\|_{m+1} < c'$ which would lead to equivalence, i.e. to $X_m = X_{m+1}$.
Next we shall give an explicit construction for an element x which satisfies the requirements of the lemma. We put $x = x_1 + x_2 + \ldots$ and determine elements $x_1, x_2, \ldots$ for which

$$\|x_1\|_1 > c_1 + 1 ;$$

$$\|x_2\|_1 < \tfrac{1}{2} \qquad \|x_2\|_2 > c_2 + 1 + \|x_1\|_2 ;$$

$$\|x_3\|_2 < \tfrac{1}{2^2} \qquad \|x_3\|_3 > c_3 + 1 + \|x_1\|_3 + \|x_2\|_3 ;$$

etc.

Then it is clear that the series $\sum_{k=1}^{\infty} x_k$ converges for all norms.

Further we have

$$\|x\|_m \geqq \|x_m\|_m - \sum_{k=m+1}^{\infty} \|x_k\|_m - \sum_{k=1}^{m-1} \|x_k\|_m \geqq$$

$$\geqq \left\{ c_m + 1 + \|x_1\|_m + \ldots + \|x_{m-1}\|_m \right\} - 1 - \left\{ \|x_1\|_m + \ldots + \|x_{m-1}\|_m \right\} \geqq c_m$$

q.e.d.

At the same time we have now proved that the topology of a s.n.s. obtained by a sequence of non-equivalent norms, for which the completions X_m $(m=1,2,\ldots)$ are all different, is essentially different from that of a normed space. It is easily seen that it is already sufficient that the sequence $\|x\|_m$ $(m=1,2,\ldots)$ contains a subsequence of non-equivalent norms. On the other hand a sequence of norms for which after a certain index all norms are equivalent clearly leads to a s.n.s. which is merely a normed space.

7. Functionals on a sequentially normed space

In section 3 we studied a number of properties of continuous linear functionals (f,x) on a linear topological space X. In section 5 the general theory was applied to the important special case where X is a normed space. For a clear understanding of the foundations of the theory of generalized functions or distributions we need some detailed knowledge on continuous linear functionals on a sequentially normed space which is somewhat between the too general linear topological space and the too special normed space.

However, as we shall see below a number of properties of functionals on a s.n.s. are closely analogous to those of functionals on a normed space.

It will be assumed that X is a complete s.n.s. with an increasing sequence of concordant norms

(7.1) $$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \dots$$

A base of zero neighbourhoods is formed by the spheres

(7.2) $$\|x\|_m < \varepsilon \qquad (m=1,2,\dots, \quad \varepsilon > 0) .$$

Convergence $x_n \to 0$ in X means that

(7.3) $$\lim_{n\to\infty} \|x_n\|_m = 0 \qquad \text{for each } m.$$

If X_m denotes the completion of X with respect to the norm $\|x\|_m$ we know by theorem 6.1 that

(7.4) $$X_1 \supset X_2 \supset \dots \supset X_m \supset \dots, \qquad \lim X_m = X.$$

We repeat the definition of a continuous linear functional.

1° (f,x) is linear if
$$(f, \alpha x+\beta y) = \alpha(f,x) + \beta(f,y).$$

2° (f,x) is continuous if for each $\varepsilon > 0$ there exists a zero neighbourhood U such that $|(f,x)| < \varepsilon$ for all $x \in U$.

As we have seen in section 3 for a linear topological space any continuous linear functional is bounded on some neighbourhood of zero. Since for a s.n.s. a base at zero is formed by the spheres $\|x\|_1 < \varepsilon$, $\|x\|_2 < \varepsilon$, ..., $\|x\|_m < \varepsilon$, ... this means that for any continuous linear functional there is an index m and a constant C such that

(7.5) $$|(f,\varphi)| \leq C \|x\|_m .$$

Letus assume that each space $X^{(m)}$ satisfies the first axiom of countability. Then it can be stated that a linear operator is continuous if and only if it is bounded.

Theorem 10.3

For a linear operator T on the union $X^{(\omega)}$ of s.n.s.'s in which the first axiom of countability is satisfied continuity and boundedness are equivalent.

Proof

It is sufficient to show that T transforms any arbitrary $X^{(m)}$ in some $Y^{(p)}$ for then we may apply theorem 8.1. First we note that in a linear topological space in which the first axiom of countability is satisfied for any sequence of elements x_n $(n=1,2,\ldots)$ it is possible to find numbers λ_n such that $\lambda_n x_n \to 0$. In fact, let $U_1 \supset U_2 \supset \ldots$ be a decreasing base of zero neighbourhoods then it is sufficient to take λ_n such that $\lambda_n x_n \in U_n$.

Let us suppose that the linear operator T in $X^{(\omega)}$ does not transform some $X^{(m)}$ into some $Y^{(p)}$, i.e. $TX^{(m)} \not\subset Y^{(p)}$ for no p. Then we may construct a sequence $x_n \in X^{(m)}$ $(n=1,2,\ldots)$ such that $Tx_n \notin Y^{(n)}$.

Let the numbers λ_n be chosen such that $\lambda_n x_n \to 0$. If T is continuous then it follows that also $T(\lambda_n x_n) = \lambda_n Tx_n \to 0$. However, this implies that all Tx_n belong to some $Y^{(p)}$ contrary to the assumption. If T is bounded then the elements $\lambda_n Tx_n$ are also bounded. However, also this implies that they all belong to some $Y^{(p)}$ contrary to the assumption.

For a continuous linear operator T transforming a space $X^{(w)}$ into a similar space $Y^{(\omega)}$ we may determine a conjugate operator T^* which transforms $Y^{(\omega)'}$ into $X^{(\omega)'}$ according to

$$(T^* g, x) = (g, Tx)$$

where $x \in X^{(\omega)}$ and $g \in Y^{(\omega)'}$. In virtue of the continuity of g and T also the conjugate operator T^* is continuous.

duced.

The sequence of functionals $f_n \in X^{(\omega)'}$ is said to converge to the functional f if for each $x \in X^{(\omega)}$

$$\lim_{n\to\infty} (f_n, x) = (f, x).$$

This type of convergence might be called weak convergence. We shall, however, not introduce any other type of convergence.

<u>Theorem 10.1</u>

If all $X^{(m)}$ are sequentially normed spaces then $X^{(\omega)'}$ is complete with respect to its (weak) convergence.

<u>Proof</u>

Let the functional f be defined by $(f,x)=\lim_{n\to\infty} (f_n,x)$ then obviously f is linear. According to theorem 7.5 for each $X^{(m)}$ it is continuous. Then by virtue of the definition it is also continuous in $X^{(\omega)'}$.

The set S of $X^{(\omega)'}$ is said to be <u>bounded</u> if for arbitrary $x \in X^{(\omega)}$ the numbers (f,x), $f \in S$ are bounded.

As regards the principle of uniform boundedness we may prove the following theorem

<u>Theorem 10,2</u>

If all $X^{(m)}$ are sequentially normed spaces and if S is a bounded set of $X^{(\omega)'}$ then the numbers (f,x), $f \in S$ are uniformly bounded on an arbitrary bounded set $A \subset X^{(\omega)}$.

<u>Proof</u>

Using the definition of boundedness in $X^{(\omega)}$ there is an index m such that $A \subset X^{(m)}$ and that A is bounded with respect to its topology. The functionals $f \in S$ are in particular continuous on $X^{(m)}$ and since they are bounded for each element $x \in X^{(m)}$ they form a weakly bounded set in $X^{(m)'}$. But then according to theorem 7.4 they are also strongly bounded, i.e. $\sup |(f,x)| < \infty$ for $f \in S$ and $x \in A$.

The linear operator T transforming a union of s.n.s. $X^{(\omega)}$ into itself or into a similar space $Y^{(\omega)}$ is said to be <u>continuous</u> if $x_n \to 0$ $(n=1,2,\dots)$ implies $Tx_n \to 0$. It is said to be <u>bounded</u> if it transforms every bounded subset of $X^{(\omega)}$ into a bounded subset of $Y^{(\omega)}$.

10. Union of sequentially normed spaces

Let there be given an increasing sequence of linear topological spaces

$$X^{(1)} \subset X^{(2)} \subset \ldots \subset X^{(m)} \subset \ldots$$

It will be assumed that at each inclusion convergence is preserved, i.e. if the sequence $x_n \in X^{(m)}$ converges to zero then it also converges to zero in the larger space $X^{(m+1)}$.

The union of all $X^{(n)}$ $(n=1,2,\ldots)$ will be indicated by $X^{(\omega)}$. It represents a linear space with the usual linear relations.

We shall introduce in $X^{(\omega)}$ the following type of convergence. The sequence x_n $(n=1,2,\ldots)$ of elements of $X^{(\omega)}$ is said to converge to the limit x of $X^{(\omega)}$ if all elements x_n and x belong to some subspace $X^{(m)}$ and if $x_n \to x$ in the topology of $X^{(m)}$.

The union $X^{(\omega)}$ of the linear topological spaces $X^{(n)}$ will not be considered as a topological space and no definition of open and closed sets in $X^{(\omega)}$ will be given.

Example

We consider the union K of the spaces K(m) consisting of all infinitely differentiable functions $\varphi(x)$ which vanish outside (-m,m). Convergence $\varphi_n \to \varphi$ in K means that all φ_n and φ vanish outside some interval (-m,m) with a fixed m and that $\varphi_n(x)$ and each derivative converges to $\varphi(x)$ and the corresponding derivative uniformly in (-m,m).

The set $S \subset X^{(\omega)}$ is said to be bounded if there is an index m such that $S \subset X^{(m)}$ and that S is bounded in the topology of $X^{(m)}$.

The linear functional f on $X^{(\omega)}$ is said to be continuous if it is continuous on each $X^{(m)}$. If each space $X^{(m)}$ satisfies the first axiom of countability this definition is equivalent to the following:

The linear functional f is continuous if it is bounded on each bounded set of $X^{(\omega)}$.

All continuous linear functionals on $X^{(\omega)}$ form the conjugate space $X^{(\omega)'}$ which, of course, is a linear space. In this space the following type of convergence will be introduced-

Proof

Remembering that $X_1 \supset X_2 \supset \ldots \supset X$ there are two possibilities. Either all X_n are separable or at least one is not separable. In the first case a dense set S in X is obtained as the (countable) union of dense sets S_n in X_n $(n=1,2,\ldots)$. In the second case we may assume e.g. that X_1 is not separable. According to the axiom of choice we may imagine a not countable subset Z_1 of X which is bounded with respect to the norm of X_1 and for which the distance of any two points always exceeds some positive number ε. Z_1 may not be bounded with respect to the norm $\|\cdot\|_2$ but then we can take a bounded subset Z_2 of Z_1 which is equally not countable.

Proceeding in this way a nesting sequence of not countable sets Z_n is obtained of which Z_n is bounded with respect to the norm $\|\cdot\|_n$. We note that for any two points of Z_1 $\|x'-x''\|_n > \varepsilon$ for all n.

Now we take a sequence x_n $(n=1,2,\ldots)$ by taking for x_n an arbitrary element of Z_n. This sequence is bounded in the topology of X but clearly it does not contain any converging subsequence, However, this contradicts the property of compactness.

From the proceding theorems we may at once deduce.

Corollary

In the conjugate space of a perfect space each bounded set is compact in the weak and strong sense.

2 Let $f_n \to 0$ in X'. Then according to theorem 7.3 there is an index l such that $f_n \in X'_l$ and that the sequence is bounded with respect to its norm. We take the index $m > l$ such that the boundedness of every sequence $x_n \in X$ with respect to the norm $\|\cdot\|_m$ implies its compactness with respect to the norm $\|\cdot\|_l$. The sequence f_n which is bounded for the norm $\|\cdot\|_l$ is also bounded for each higher norm and in particular for $\|\cdot\|_m$. We shall show that $f_n \to 0$ with respect to the latter norm, i.e. $(f_n, x) \to 0$ uniformly in the unit sphere $\|x\|_m \leq 1$. If this were not true it would be possible to construct a sequence x_n such that with some positive number ε

$$\|x_n\|_m \leq 1 \quad (n=1,2,\dots) , \quad |(f_n, x_n)| \geq \varepsilon .$$

The sequence x_n is compact for the norm $\|\cdot\|_l$, say that $x_n \to x$ in X_m. But then

$$|(f_n, x_n)| \leq |(f_n, x_n - x)| + |(f_n, x)| \leq \|f_n\|_l \, \|x_n - x\|_l + |(f_n, x)|,$$

so that with $n \to \infty$ a contradiction is obtained.

The conjugate of a perfect space is in general not perfect, it is not even a sequentially normed space. However, it can be shown that just as a perfect space it enjoys the property that its bounded sets are compact.

Theorem 9.4

In the conjugate space X' of a separable s.n.s. X each bounded sequence $f_n (n=1,2,\dots)$ contains a weakly convergent subsequence.

Proof

Using the diagonal process a subsequence $f_{n_1}, f_{n_2}, \dots$ can be constructed which converges for each element x_k $(k=1,2,\dots)$ of a dense set S in X. The chosen subsequence is bounded and belongs to X'_m for some index m (theorem 7.3). The functionals $f_{n_1}, f_{n_2}, \dots$ converge on S which is dense in X and hence dense in X_m. Then (corollary 2 of theorem 7.4) they are weakly convergent to some limit $f \in X'_m$. This means that for any $x \in X_m$, and in particular for any $x \in X$, $(f_{n_k}, x) \to (f, x)$ q.e.d.

This theorem holds in particular for a perfect space, for we have

Theorem 9.5

A perfect space is separable.

Then it will be shown that S is compact with respect to the norm $\|\varphi\|_{m-1}$. The derivatives of $\varphi^{(m-1)}(x)$ $(\varphi \in S)$ are uniformly bounded so that according to the theorem 4.3. of Arzela-Ascoli it is possible to select a uniformly convergent sequence $\varphi_1^{(m-1)}(x)$, $\varphi_2^{(m-1)}(x), \ldots$. Then also the sequences $\varphi_1^{(k)}(x)$, $\varphi_2^{(k)}(x), \ldots$ $(k=0,1,\ldots,m-2)$ are uniformly convergent. This means that the sequence $\varphi_n(x)$ converges with respect to the norm $\|\varphi\|_{m-1}$. Hence, according to the theorem K(a) is perfect.

Theorem 9.2

In the conjugate X' of a perfect space X strong and weak convergence are equivalent.

Proof

It is sufficient to show that $f_n \to 0$ in the weak sense implies $f_n \to 0$ in the strong sense, i.e. $(f_n, x) \to 0$ uniformly on every bounded subset A of X. It follows from the first corollary of theorem 7.4 that the set f_n $(n=1,2,\ldots)$ is already strongly bounded. If (f_n,x) is not uniformly convergent in a bounded set A it is possible to construct a sequence x_n $(n=1,2,\ldots)$ such that $|(f_n,x_n)| > \varepsilon$ for some positive number ε. On the other hand the sequence x_n has a limit point x since A is compact. Let us assume that $x_n \to x$. Then according to theorem 7.7 we have $(f,x_n) \to (f,x)$ uniformly in every bounded set of X'.
If for the latter set we take the elements f_n then we have in particular $(f_n, x_n - x) \to 0$. But since also $(f_n,x) \to 0$ we would also have $(f_n,x_n) \to 0$ thereby obtaining a contradiction.

For perfect spaces in which the conditions of theorem 9.1 are satisfied something more may be said concerning (weak or strong) convergence in the conjugate space.

Theorem 9.3

With the conditions of theorem 9.1 the sequence f_n $(n=1,2,\ldots)$ converges (weakly and strongly) in X' if and only if there is an index m such that all f_n belong to X'_m and are convergent with respect to its norm.

Proof

1 If $f_n \in X'_m$ and $\|f_n - f\|_m \to 0$ then for each $x \in X$ we have $|(f_n - f, x)| \leq \|f_n - f\|_m \|x\|_m \to 0$ so that f_n is (weakly) convergent to f in X'.

9. Perfect spaces

A sequentially space with the property that all its bounded subsets are compact is said to be a perfect space. We recall the property enjoyed by any linear topological space that any compact set is also bounded (theorem 3.2). A simple criterion that a given s.n.s. be perfect is:

Theorem 9.1

Let X be a sequentially normed space with norms

$$\|x\|_1 \leq \|x\|_2 \leq \dots \leq \|x\|_m$$

and let there be given a sequence of indices $m_1 < m_2 < \dots < m_k < \dots$. Then if from every set $S \subset X$ which is bounded with respect to the norm $\|\cdot\|_{m_{k+1}}$ it is possible to select a sequence with distinct elements which is fundamental with respect to the norm $\|\cdot\|_{m_k}$ the space X is perfect.

Proof

Let S be a bounded subset of X. In particular S is bounded with respect to $\|\cdot\|_{m_2}$. It contains a sequence $x_{11}, x_{12}, \dots$ which is fundamental with respect to $\|\cdot\|_{m_1}$. This sequence is further bounded with respect to $\|\cdot\|_{m_3}$ and contains a subsequence $x_{21}, x_{22}, \dots$ which is fundamental with respect to $\|\cdot\|_{m_2}$. Continuing this procedure a system of sequences is obtained

$$\begin{matrix} x_{11} & x_{12} & \dots & x_{1m} & \dots \\ x_{21} & x_{22} & \dots & x_{2m} & \dots \\ \dots & \dots & \dots & \dots & \dots \\ x_{m1} & x_{m2} & \dots & x_{mm} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix}$$

each of which is fundamental to the previous norm. The diagonal sequence $x_{11}, x_{22}, \dots$ is fundamental with respect to all norms i.e. fundamental in the topology of X. Since X is complete the limit exists so that S is compact.

Example The space K(a) is perfect

In fact, let S be a bounded set with respect to the norm

$$\|\varphi\|_m = \max\left\{|\varphi(x)|, \dots, |\varphi^{(m)}(x)|\right\}, \quad m \geq 1.$$

For a sequence T_n (n=1,2,...) we may introduce weak convergence in the usual way as follows:

A sequence T_n (n=1,2,...) of continuous linear operators is said to be <u>weakly convergent</u> to the limit T if

$$\lim T_n x = T x \qquad \text{for all } x \in X$$

in the topology of X^*.

It is not a priori obvious that the limit operator is linear and continuous. Clearly it is linear; that it is also continuous is expressed in the following theorem.

<u>Theorem 8.4</u>

The limit T of a weakly convergent sequence of continuous linear operators T_n mapping a s.n.s. X into a s.n.s. X^* is also linear and continuous.

<u>Proof</u>

In view of theorem 8.1 it is sufficient to prove that T transforms every bounded set $A \subset X$ into a bounded set TA of X^*. In order to prove the boundedness of TA we may use theorem 7.2. Hence there remains to show that for each continuous linear functional g the numbers $|(g,Tx)|$, $x \in A$ are (uniformly) bounded.

With a fixed g the sequence of continuous linear functionals f_n defined by $(f_n,x) = (g,T_n x)$ (n=1,2,...) is weakly convergent to the limit $(f,x) = (g,Tx)$.

According to theorem 7.5 the limit f is also continuous. However, this means that $\sup|(f,x)| < \infty$ for $x \in A$. But then also $\sup|(g,Tx)| < \infty$ for $x \in A$ which proves the theorem.

We note that this theorem rests essentially upon the principle of uniform boundedness (see theorem 5.8).

Using the concept of concordant topologies the following theorem may be formulated

Theorem 8.3

If X and $X^* \subset X$ are (complete) sequentially normed spaces with concordant topologies then convergence $x_n \to x$ in the topology of X^* implies convergence $x_n \to x$ in the topology of X.

Proof

Let $\|x\|_m$ and $\|x\|'_m$ (m=1,2,...) be the norm systems of X and X^*. Then we introduce a third norm system by means of $\|x\|''_m = \max\{\|x\|_m, \|x\|'_m\}$. First we shall show that X^* is complete with respect to the new norm. In fact, let $x_n \in X^*$ (n=1,2,...) be a fundamental sequence for $\|x\|''_m$. Then it is also fundamental for $\|x\|_m$ and $\|x\|'_m$ so that $x_n \to x_o$ in X and $x_n \to x_o^*$ in X^*. From the concordance of the topologies it follows easily that $x_o = x_o^*$. Hence also $x_n \to x_o$ for the new norm.
From the corollary of theorem 8.2 it follows that in X^* the comparable norms $\|x\|'_m$ and $\|x\|''_m$ are equivalent.
Let now $x_n \to x_o$ in the topology of X^*. This means of course that $\|x_n - x_o\|'_m \to 0$ (m=1,2,...). The equivalence proved above says that then $\|x_n - x_o\|''_m \to 0$. But this implies $\|x_n - x_o\|_m \to 0$ i.e. $x_n \to x_o$ in the topology of X.

This theorem means that with the condition of concordant norm systems convergence in the narrower space implies convergence in the wider space. Also we have

Corollary

Any continuous linear functional on the wider space X is also a continuous linear functional on the narrower space X^*.

In section 1 we have seen that the linear (continuous) operators $T(X \to X^*)$ where X and X^* are arbitrary linear topological spaces form a linear space. In fact, the following definitions of addition and scalar multiplication

$$(T_1 + T_2)x = T_1 x + T_2 x,$$

$$(\lambda T)x = \lambda Tx,$$

where $x \in X$,
clearly satisfy the axioms of the linear space. Moreover, it is easily seen that the continuity of T_1 and T_2 implies that of $T_1 + T_2$ and the continuity of T that of λT.

8. Operators on a sequentially normed space

We consider a continuous linear operator T which transforms a s.n.s. X into a s.n.s. X^*.
The general theory of a continuous linear operator on a linear topological space has been given in section 3. According to the theorems 3.3^a and 3.3^b the continuity of a linear operator on a s.n.s. is equivalent with boundedness. The theory of linear operators was continued in section 4 where we discussed Banach's theorem of the inverse operator with respect to a complete linear metric space. Since a s.n.s. is also a linear metric space this important result also holds for a s.n.s.

We repeat the definition of a continuous linear operator.
1^o Tx is linear if

$$T(\alpha x+\beta y) = \alpha Tx + \beta Ty.$$

2^o Tx is continuous if for each neighbourhood V of Tx there is a neighbourhood U of x such that $TU \subset V$.

Next we repeat the following theorem

Theorem 8.1

A linear operator is continuous if and only if it transforms bounded sets into bounded sets.

Banach's theorem 4.6 may be formulated as follows

Theorem 8.2

A continuous linear operator T which transform a (complete) s.n.s. X one-to-one into a s.n.s. X^* has a continuous inverse T^{-1}.

As in section 6 we have the following corollary

Corollary

If X is a (complete) s.n.s. with respect to two sequences of norms: $\|x\|_m$ and $\|x\|'_m$ (m=1,2,...) and if these sequences are comparable then they are equivalent.

Let X and X^* be two linear topological spaces where X^* is a subspace of X; then the topologies of X and X^* are said to be concordant if for each sequence $x_n \in X^*$ which converges to zero in the topology of X^* and which at the same time converges to an element x in the topology of X we always have x=0. A common situation in which this arives is that where the norms in X^* are stronger than those in X.

Again the sequence f_n is strongly bounded. We now apply theorem 7.3 which says that all f_n belongs to some X'_m and are bounded there. Since $(f_n,x)\to(f,x)$ on X which is dense in X_m convergence also holds in X_m.

$\underline{2}^o$ If $f_n \to f$ for $x \in X_m$ then convergence certainly holds in the subspace X of X_m.

Later on we shall need the following lemma

Theorem 7.7

If the sequence x_n $(n=1,2,\dots)$ converges to the element x of a s.n.s. X then $(f,x_n)\to(f,x)$ uniformly for every bounded subset of functionals f of X'.

Proof

We note that by theorem 7.4 it makes no difference whether bounded is meant in the strong or in the weak sense. Let B be a bounded subset of X'; then by theorem 7.3 we know that there exists an index m such that $B \subset X'_m$ and that B is bounded with respect to its norm, say

$$\|f\|_m < M , \quad f \in B.$$

For each n we have

$$\sup_{f \in B} |(f,x_n-x)| \leq \sup_{f \in B} \|f\|_m \; \|x_n-x\|_m < M \; \|x_n-x\|_m .$$

Hence $(f,x_n-x)\to 0$ uniformly in B.

Corollary 1

If f_n is a weakly convergent sequence of continuous linear functionals on a s.n.s. then f_n is strongly bounded.

Corollary 2

The sequence f_n of continuous linear functionals on a s.n.s. is weakly convergent to zero if and only if the sequence is strongly (weakly) bounded and if $(f_n,x)\to 0$ at least for those x belonging to a set which is dense in X.

We are now able to prove the following important theorem.

Theorem 7.5

The conjugate space of a sequentially normed space is complete with respect to weak convergence.

Proof

We consider a weakly fundamental sequence $f_n \in X'$ $(n=1,2,\ldots)$. This means that for each $x \in X$ the sequence (f_n,x) converges to some limit, say (f,x). The limit is obviously a linear functional. There remains to prove its continuity. We have just shown (Corollary 1 of the preceding theorem) that the sequence f_n is also strongly bounded. Then according to lemma 3.9 there exists a neighbourhood of zero U on which the functionals f_n are bounded:

$$|(f_n,x)| \leq C, \qquad x \in U.$$

But then also

$$|(f,x)| = \lim_{n\to\infty} |(f_n,x)| \leq C, \qquad x \in U.$$

This means that f is bounded on U which according to e.g. theorem 3.7 implies continuity.

Theorem 7.6

The sequence f_n of continuous linear functionals on a s.n.s. converges weakly to the functional f if and only if all f_n are continuous functionals on a common normed space X_m and if they are weakly convergent in X_m i.e.

$$(f_n,\ x) \to (f,x), \qquad x \in X_m.$$

Proof

1^0 Let f_n be weakly convergent to f.

index m the set S is contained in X'_m and bounded with respect to its norm.

<u>Proof</u>

<u>1</u>^{o} Let $S \subset X'_m$ and be bounded in X'_m.
Take the zero neighbourhood $U \subset X$ with $\|x\|_m \leq 1$. The boundedness of S in X'_m means that

$$|(f,x)| \leq M \quad \text{for} \quad x \in U, \quad f \in S.$$

But this means that S is bounded on some zero neighbourhood of X. Hence S is bounded on every bounded set in X, i.e. strongly bounded.

<u>2</u>^{o} Let S be strongly bounded.
According to lemma 3.9 there exists some zero neighbourhood U of X, say $\|x\|_m < \varepsilon$, for which

$$\sup_{f \in S} |(f,x)| < M$$

for some constant M. But this means that every functional $f \in S$ is bounded on U by the constant M so that $f \in X_m$ with a norm not exceeding M/ε q.e.d.

It will be clear that strong convergence implies weak convergence and that any strongly bounded set is also weakly bounded. However, as for Banach spaces (theorem 5.8) we have

<u>Theorem 7.4</u>

A weakly bounded set of X' is also strongly bounded.

<u>Proof</u>

It is sufficient to show that a weakly bounded set S is bounded on some zero neighbourhood of X. Consider the set C of all $x \in X$ for which

$$(7.13) \qquad |(f,x)| \leq 1 \qquad \text{for all} \quad f \in S.$$

The set C is closed since for each $f \in S$ the set for which $|(f,x)| \leq 1$ is closed. The set C is convex since for each $f \in S$ the set for which $|(f,x)| \leq 1$ is convex. Moreover, C is symmetric. Finally C absorbs all elements of X. In fact, since S is weakly bounded the numbers $|(f,x)|$, $f \in S$ are bounded for any fixed x so that e.g. $|(f,x_o)| \leq M$ and hence $|(f,x_o/M)| \leq 1$, i.e. $x_o/M \in C$. Now we may apply theorem 4.2. Hence C contains a zero neighbourhood U. On this neighbourhood S is uniformly bounded in view of (7.13). This proves the theorem.

$\|\cdot\|_m$ for arbitrary m. We may consider S as a subset of X_m. Then on S in particular all continuous linear functionals of the order m, which constitute X_m', are bounded.
But then theorem 5.8 can be applied.

According to the general theory of section 3 in the conjugate space X' a weak topology and a strong topology can be introduced.

The weak topology is determined by weak zero neighbourhoods depending on a finite number of elements $x_1, x_2, \ldots, x_m$ of X and a positive number ε. The neighbourhood $U(x_1, x_2, \ldots, x_m, \varepsilon)$ is defined as the set of those $f \in X'$ for which

$$|(f,x_1)| < \varepsilon, \; |(f,x_2)| < \varepsilon, \ldots, |(f,x_m)| < \varepsilon. \tag{7.9}$$

The set $S \subset X'$ is weakly bounded if for each $x \in X$

$$\sup_{f \in S} |(f,x)| < \infty . \tag{7.10}$$

The sequence $f_n \in X'$ $(n=1,2,\ldots)$ is weakly convergent to f if for each $x \in X$

$$(f_n, x) \to (f,x).$$

The strong topology is determined by strong zero neighbourhoods depending on a bounded set B of X and a positive number ε. The neighbourhood $U(B, \varepsilon)$ is defined as the set of those $f \in X'$ for which

$$\sup_{x \in B} |(f,x)| < \varepsilon. \tag{7.11}$$

The set $S \subset X'$ is strongly bounded if for every bounded subset A of X

$$\sup_{x \in A,\; f \in S} |(f,x)| < \varepsilon. \tag{7.12}$$

The sequence $f_n \in X'$ $(n=1,2,\ldots)$ is strongly convergent to f if

$$(f_n, x) \to (f,x)$$

uniformly in every bounded subset B of X.

The notion of a strongly bounded set may here be put in a simpler form.

Theorem 7.3
The set S of X' is strongly bounded if and only if for some

differentiable functions $\varphi(x)$ which vanish outside $(-a,a)$.
We have seen that $K(a)$ is a complete s.n.s. with the norms

$$\|\varphi\|_m = \max_{|x|\leq a} \left\{ |\varphi(x)|, |\varphi'(x)|, \ldots, |\varphi^{(m)}(x)| \right\},$$

$m=0,1,2,\ldots$.

We have seen that every continuous linear functional on $K(a)$ turns out to be a continuous linear functional on the normed space $K_m(a)$ for some m. The space $K_m(a)$ consists of the functions $\varphi(x)$ which have continuous derivatives up to the m^{th} order and which vanish outside $(-a,a)$. To any $\varphi \in K_m(a)$ we may associate the continuous function $\psi(x)= \varphi^{(m)}(x)$. This defines a mapping of $K_m(a)$ into a subspace of $C(a)$, the space of all continuous functions in $(-a,a)$. It is easily seen that this is a continuous one-to-one mapping since

$$\|\varphi\|_m = \max \left\{ |\varphi|, |\varphi'|, \ldots, |\varphi^{(m)}| \right\} \leq C \max |\varphi^{(m)}| =$$
$$= C \max |\psi| \leq C \|\varphi\|_m;$$

or more constructively

$$\varphi(x) = \int_{-a}^{x} \varphi' d\xi = \ldots = \int_{-a}^{x} \int_{-a}^{\xi_m} \ldots \int_{-a}^{\xi_2} \varphi^{(m)}(\xi_1) d\xi_1 d\xi_2 \ldots d\xi_m .$$

Therefore (f,φ) is equivalent to a continuous linear functional $(g,\psi) = (f,\varphi)$ on the subspace of $C(a)$. According to the theorem 5.6 of Hahn-Banach this functional can be extended to the whole space $C(a)$. Then we may apply the representation theorem 5.10 of Riess saying that there exists a function of bounded variation $\mu(x)$ for which

$$(g,\psi) = \int_{-a}^{a} \psi(x) \, d\mu(x).$$

Hence we obtain

$$(f,\varphi) = \int_{-a}^{a} \varphi^{(m)}(x) \, d\mu(x). \tag{7.8}$$

For sequentially normed spaces the following analogue of theorem 5.9 may be formulated (principle of uniform boundedness).

Theorem 7.2

If S is a subset of a s.n.s. X then S is bounded if for each continuous linear functional f the numbers $|(f,x)|$, $x \in S$ are bounded.

Proof

According to the definition of boundedness in a s.n.s. it is sufficient to prove that S is bounded with respect to the norm

This result is important enough to state it in the form of a theorem:

Theorem 7.1

Every continuous linear functional on a s.n.s. is bounded with respect to some norm in the sequence.

The converse of this theorem is obvious.
The least number m for which (7.5) is true is said to be the order of the functional.

All continuous linear functionals on a s.n.s. X form the conjugate X'. We shall now study the structure of this space.

All continuous linear functionals of an order $\leq m$, i.e. those functionals which are continuous with respect to the norm of X_m form a subspace X'_m of X' which is the conjugate of X_m. Therefore X'_m is a complete normed space. We have

$$X' = \bigcup_{k=1}^{\infty} X'_k$$

It is obvious that a functional of order m is also bounded on the spheres $\|x\|_{m+1} \leq 1$, $\|x\|_{m+2} \leq 1, \ldots$ so that it is an element of $X'_{m+1}, X'_{m+2}, \ldots$. Hence we have the nesting sequence

(7.6) $$X'_1 \subset X'_2 \subset \ldots \subset X'_m \subset \ldots, \quad \lim X'_m = X.$$

The functional f of order m has in $X'_m, X'_{m+1}, \ldots$ the following norms

$$\|f\|_m = \sup_{\|x\|_m=1} |(f,x)|, \quad \|f\|_{m+1} = \sup_{\|x\|_{m+1}=1} |(f,x)|, \ldots$$

so that

(7.7) $$\|f\|_m \geq \|f\|_{m+1} \geq \ldots$$

This may be summarized as follows.

Property

The space X' which is the conjugate of the s.n.s. X is the union of an increasing sequence of complete normed spaces with norms becoming weaker and weaker.

Example

It is now possible to obtain the general form of a continuous linear functional (f,φ) on the space K(a) of all infinitely